ON THE STABILITY OF AVERAGE EFFECTIVE WAGE RATE IN ENTERPRISE

Andrej Balciar

Dept. of Quantitative Methods and Informatics, The Faculty of Economics University of Matej Bel,
Tajovského 10, 975 92 Banská Bystrica, Slovakia
e-mail: andrej.balciar@umb.sk

Summary In the article a model of the investment activity of an enterprise including financing of investment from new loans, from net profit and with the help of depreciation is designed. Equilibrium of average effective wage rate is defined and its stability is examined.

1. PRELIMINARIES

Consider the neoclassical production function $F(.)$ with 2 production factors - labour and capital:

$$Y(t)=F\left(K(t),\hat{L}(t)\right) \tag{1}$$

where $Y(t)$ - value added of an enterprise, $K(t)$ - capital of an enterprise, $e^{x.t}$ ($x>0$) - the rate of labour-augmenting technological progress, $L(t)$ - the amount of labour in an enterprise, $\hat{L}(t)=L(t)e^{xt}$ - effective labour, t - time.

We will assume that the number of working hours is increasing by the rate n:

$$\frac{L'}{L}=n \tag{2}$$

Assume as in [1] that the neoclassical production function $F(.)$ satisfies the following conditions:

$$\frac{\partial F}{\partial K}>0,\ \frac{\partial F}{\partial L}>0,\ \frac{\partial^2 F}{\partial K^2}<0,\ \frac{\partial^2 F}{\partial L^2}<0, \tag{3}$$

$$\forall\lambda>0;\ F\left(\lambda K,\lambda\hat{L}\right)=\lambda F\left(K,\hat{L}\right), \tag{4}$$

$$\lim_{K\to 0}F_K=\lim_{L\to 0}F_L=\infty,\ \lim_{K\to\infty}F_K=\lim_{L\to\infty}F_L=0. \tag{5}$$

We will work with the variable $\hat{k}$ - capital per effective labour:

$$\hat{k}=\frac{K}{\hat{L}}. \tag{6}$$

Because production function $F(.)$ is homogeneous of the first degree ($\lambda=\hat{L}$), we can write:

$$F\left(K,\hat{L}\right)=\hat{L}.F\left(\frac{K}{\hat{L}},1\right). \tag{7}$$

After putting (6) into (7) we have:

$$F\left(K,\hat{L}\right)=\hat{L}.F\left(\hat{k},1\right). \tag{8}$$

Utilizing denotation $f\left(\hat{k}\right)\equiv F\left(\hat{k},1\right)$ we can re-write (8):

$$F\left(K,\hat{L}\right)=\hat{L}.f\left(\hat{k}\right). \tag{9}$$

2. THE MODEL OF THE INVESTMENT ACTIVITY OF ENTERPRISE

We are going to design the model of the investment activity of enterprise. Net profit of enterprise is defined by the following formula:

$$\pi_N=(1-r_T).\left(F\left(K,\hat{L}\right)-w.L-\delta.K-r_I.D\right), \tag{10}$$

where r_T - the income tax rate, w - average wage rate, δ - the rate of depreciation, r_I - average interest rate of accepted loans, D - the debt.

In the paper we will assume the situation that the company reaches a profit. The increase in capital - K' is represented by the increase in net investments, i.e. investments - I minus depreciation of capital - δK:

$$K'=I-\delta K. \tag{11}$$

Investments - I are assumed to be financed by new loans - U, a part of net profit, and by depreciation in the form of tax shield: $r_T\delta K$. Let p_I, $p_I\in(0,1)$, be a portion of the investments on net profit. Then the investments from the financing point of view can be expressed in the form:

$$I=U+p_I\pi_N+r_T\delta K. \tag{12}$$

The increase in debt - D' is determined by the amount of new loans minus instalments of all active loans. Let s, $s\in(0,1)$, be a portion of instalments of loans on debt of the enterprise. Then for the increase in debt it holds:

$$D'=U-s.D. \tag{13}$$

We assume that the enterprise pursues a goal of holding debt-to-capital ratio fixed. Let d, $d\in(0,1)$, be debt-to-capital ratio, then we can write:

$$D = d.K \,. \quad (14)$$

In the paper we will consider a Cobb-Douglas production function:

$$F\left(K,\hat{L}\right) = K^{\alpha}.\hat{L}^{1-\alpha} \,. \quad (15)$$

where α, $\alpha \in (0,1)$, is the elasticity of the capital change with respect to the change of value added. Utilizing (9) and (6) in Cobb-Douglas production function we obtain:

$$f\left(\hat{k}\right) = K^{\alpha}.\hat{L}^{-\alpha} = \hat{k}^{\alpha} \,. \quad (16)$$

Cobb-Douglas production function can be written in the form:

$$F\left(K,\hat{L}\right) = F\left(K,L,t\right) = K^{\alpha}.L^{1-\alpha}.e^{x(1-\alpha).t} \,. \quad (17)$$

Under the assumption of perfect competition in the labour market marginal productivity of labour determines average wage rate - w:

$$\frac{\partial F\left(K,\hat{L}\right)}{\partial L} = w \,. \quad (18)$$

We will use (17) to determine marginal productivity of labour:

$$\frac{\partial F\left(K,L,t\right)}{\partial L} = \left(1-\alpha\right) K^{\alpha}\left(Le^{x.t}\right)^{-\alpha} e^{x.t} \,,$$

$$\frac{\partial F\left(K,L,t\right)}{\partial L} = \left(1-\alpha\right) K^{\alpha}\hat{L}^{-\alpha}e^{x.t} \,. \quad (19)$$

Putting (6) into (19) we obtain an expression for marginal productivity of labour:

$$\frac{\partial F\left(K,L,t\right)}{\partial L} = \left(1-\alpha\right)\hat{k}^{\alpha}e^{x.t} \,. \quad (20)$$

Because marginal productivity of labour determines average wage rate, from (18) and (20) we have

$$w = \left(1-\alpha\right)\hat{k}^{\alpha}e^{x.t} \,. \quad (21)$$

We will work with the variable $\hat{w}$ - average effective wage rate:

$$\hat{w} = \frac{w}{e^{x.t}} \,. \quad (22)$$

According to (21) and (22) we have:

$$\hat{w} = \left(1-\alpha\right)\hat{k}^{\alpha} \,. \quad (23)$$

Utilizing (14), (13), (12) in (11) we obtain:

$$K' = \frac{p_I.(1-r_T)}{1-d}.\left(F\left(K,\hat{L}\right) - w.L - \delta.K - r_I.d.K\right) - \frac{s.d + (1-r_T)\delta}{1-d}.K \,. \quad (24)$$

From (6) we can express capital K. Differentiating of capital with respect to time we have:

$$K' = \hat{k}'.\hat{L} + \hat{k}.\hat{L}' \,. \quad (25)$$

After differentiating of effective labour with respect to time we can write:

$$\hat{L}' = \left(Le^{x.t}\right)' = x.Le^{x.t} + L'e^{x.t} \,. \quad (26)$$

Putting (2) and effective labour $\hat{L} = Le^{xt}$ into (26) we obtain:

$$\hat{L}' = x.Le^{x.t} + L'e^{x.t} = x.Le^{x.t} + n.Le^{x.t} = \left(x+n\right).\hat{L} \,. \quad (27)$$

We will utilize (27) in (25). Then we have:

$$K' = \hat{L}.\left(\hat{k}' + \left(x+n\right).\hat{k}\right) \,. \quad (28)$$

Putting (6) and (9) into (24) we can write:

$$K' = \hat{L}.\left\{\frac{p_I.(1-r_T)}{1-d}.\left(f\left(\hat{k}\right) - w.e^{-x.t} - \delta.\hat{k} - r_I.d.\hat{k}\right) - \frac{s.d + (1-r_T)\delta}{1-d}.\hat{k}\right\} \,. \quad (29)$$

Comparing the right-hand sides of (28) and (29) we obtain:

$$\hat{k}' + A.\hat{k} = B.\left(f\left(\hat{k}\right) - w.e^{-x.t}\right) \,, \quad (30)$$

where:

$$A \equiv x + n + \frac{p_I.(1-r_T)}{1-d}.\left(\delta + r_I.d\right) + \frac{s.d + (1-r_T)\delta}{1-d} \,,$$

$$B \equiv \frac{p_I.(1-r_T)}{1-d} \,.$$

Substituting (16) and a formula for w from (21) into (30) we can write:

$$\hat{k}' + A.\hat{k} = \alpha.B.\hat{k}^{\alpha} \,. \quad (31)$$

The equation (31) is the fundamental equation of the model of the investment activity of enterprise. It is a Bernoulli differential equation. The solution of equation (31) given by the initial condition $\hat{k}\left(t_0\right) = k_0$ is

$$\hat{k}\left(t\right) = \left\{\left(\hat{k}_0^{1-\alpha} - \frac{\alpha B}{A}\right)e^{-(1-\alpha)A(t-t_0)} + \frac{\alpha B}{A}\right\}^{\frac{1}{1-\alpha}} \,. \quad (32)$$

3. THE EXISTENCE OF EQUILIBRIUM OF AVERAGE EFFECTIVE WAGE RATE

We say that a constant function which satisfies the differential equation of the variable x is equilibrium of the variable x.

Putting $\hat{k}\left(t\right) = c$, c is a constant, into (31) we have:

$$c = \left(\frac{\alpha.B}{A}\right)^{\frac{1}{1-\alpha}} \,. \quad (33)$$

If we denote equilibrium of capital per effective labour by $\hat{k}_e$, then

$$\hat{k}_e = \left(\frac{\alpha.B}{A}\right)^{\frac{1}{1-\alpha}}. \tag{34}$$

According to (23) it holds

$$\forall t \geq t_0; \hat{w}(t) = (1-\alpha)\hat{k}^{\alpha}(t). \tag{35}$$

On the basis of (23) and (34) we can define equilibrium of average effective wage rate.

Definition 3.1. Let $\hat{k}_e$ be equilibrium of capital per effective labour determined by the equation (31). Then we say that the variable

$$\hat{w}_e = (1-\alpha)\hat{k}_e^{\alpha}$$

is equilibrium of average effective wage rate.

In the sense of definition 3.1., utilizing (34), we can find equilibrium of average effective wage rate:

$$\hat{w}_e = (1-\alpha)\left(\frac{\alpha.B}{A}\right)^{\frac{\alpha}{1-\alpha}}. \tag{36}$$

4. THE STABILITY OF EQUILIBRIUM OF AVERAGE EFFECTIVE WAGE RATE

We will find out whether equilibrium of average effective wage rate given by (35) is asymptotic stable. Consider the initial condition: $\hat{w}(t_0) = \hat{w}_0$. According to (23) we assume that $\hat{w}_0 = (1-\alpha)\hat{k}_0^{\alpha}$. Putting (32), (36) into (35) we get a formula for average effective wage rate in time t given by the initial condition $\hat{w}(t_0) = \hat{w}_0$:

$$\hat{w}(t) = (1-\alpha)\left\{\left(\frac{\hat{w}_0^{\frac{1-\alpha}{\alpha}}}{1-\alpha} - \frac{\hat{w}_e^{\frac{1-\alpha}{\alpha}}}{1-\alpha}\right)e^{-(1-\alpha).A.(t-t_0)} + \frac{\hat{w}_e^{\frac{1-\alpha}{\alpha}}}{1-\alpha}\right\}^{\frac{\alpha}{1-\alpha}}. \tag{37}$$

First, we will state the definition of Liapunov stability of equilibrium of average effective wage rate (see for instance [3]).

Definition 4.1. We say that equilibrium of average effective wage rate given by (36) is stable in the Liapunov sense (briefly stable), if for all $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that average effective wage rate given by (36), satisfying the condition

$$|\hat{w}(t_0) - \hat{w}_e| < \delta,$$

exists for $t \geq t_0$ and satisfies for these t inequality

$$|\hat{w}(t) - \hat{w}_e| < \varepsilon.$$

We assume that average effective wage rate is positive. We distinguish 3 cases with respect to the relationship between $\hat{w}_0$ and $\hat{w}_e$.

(i) $0 < \hat{w}_0 < \hat{w}_e$

The relations (i) and (37) imply that average effective wage rate is an increasing function of time. Then from (i) we have:

$$\forall t \geq t_0; 0 < \hat{w}_0 < \hat{w}(t) < \hat{w}_e. \tag{38}$$

We will prove Liapunov stability of equilibrium of average effective wage rate in the sense of the definition 4.1.

P r o o f. Take a positive real number $\delta = \varepsilon$ and such initial value of average effective wage rate $\hat{w}(t_0) = \ = \hat{w}_0$ that:

$$|\hat{w}(t_0) - \hat{w}_e| < \delta \Leftrightarrow |\hat{w}_0 - \hat{w}_e| < \varepsilon. \tag{39}$$

Utilizing (38) we can write:

$$\forall t \geq t_0; |\hat{w}(t) - \hat{w}_e| = \hat{w}_e - \hat{w}(t) < \hat{w}_e - \hat{w}_0 = |\hat{w}_0 - \hat{w}_e|. \tag{40}$$

The inequalities (39) and (40) imply:

$$\forall t \geq t_0; |\hat{w}(t) - \hat{w}_e| < \varepsilon.$$

Thus, we have proven that equilibrium of average effective wage rate given by (36) is stable in the Liapunov sense.

(ii) $0 < \hat{w}_0 = \hat{w}_e$

Putting (ii) into (37) we obtain:

$$\forall t \geq t_0; 0 < \hat{w}_0 = \hat{w}(t) = \hat{w}_e. \tag{41}$$

With the help of the definition 4.1 it can be seen that equilibrium of average effective wage rate given by (36) is stable in the Liapunov sense.

(iii) $0 < \hat{w}_e < \hat{w}_0$

In this case Liapunov stability can be proven analogously as in case (i). From the above proves it follows that equilibrium of average effective wage rate given by (36) is stable in the Liapunov sense.

Now, we will state the definition of asymptotic stability of equilibrium of average effective wage rate (see for example [3]).

Definition 4.2. We say that equilibrium of average effective wage rate given by (36) is asymptotic stable, if it is stable and for all $t_0 \geq 0$ there exists $\varDelta = \varDelta(t_0) > 0$ such that for average effective wage rate given by (37), satisfying the condition $|\hat{w}(t_0) - \hat{w}_e| < \varDelta$, it holds

$$\lim_{t\to\infty} |\hat{w}(t) - \hat{w}_e| = 0.$$

We will prove that equilibrium of average effective wage rate given by (36) is asymptotic stable in the sense of the definition 4.2.

P r o o f. We have proven that equilibrium of average effective wage rate given by (36) is stable. Take a positive real number $\Delta = \varepsilon$ and such initial value of average effective wage rate $\hat{w}(t_0) = \hat{w}_0$ that:

$$\left|\hat{w}(t_0) - \hat{w}_e\right| < \Delta = \varepsilon . \tag{42}$$

Utilizing (37) we can count the following limit:

$$\lim_{t\to\infty} \left|\hat{w}(t) - \hat{w}_e\right| = \lim_{t\to\infty} \left|(1-\alpha)\left(\frac{\hat{w}_e^{\frac{1-\alpha}{\alpha}}}{1-\alpha}\right)^{\frac{\alpha}{1-\alpha}} - \hat{w}_e\right| =$$

$$= \lim_{t\to\infty} \left|\hat{w}_e - \hat{w}_e\right| = 0 . \tag{43}$$

Thus, we have proven that equilibrium of average effective wage rate given by (36) is asymptotic stable.

5. CONCLUSION

In the paper we have introduced the model of the investment activity in the enterprise. Equilibrium of average effective wage rate has been defined. It was proven that equilibrium of average effective wage rate is asymptotic stable.

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